Coarse-graining complex dynamics: Continuous Time Random Walks vs. Record Dynamics.

Paolo Sibani[1]

[1] FKF, University of Southern Denmark, Campusvej 55, DK5230, Odense M.

PACS 02.50.Ey – Stochastic processes
PACS 89.75.Da – Systems obeying scaling laws
PACS nn.mm.xx – Fluctuation phenomena, random processes, and Brownian motion

Abstract – Continuous Time Random Walks (CTRW) are widely used to coarse-grain the evolution of systems jumping from a metastable sub-set of their configuration space, or trap, to another via rare intermittent events. The multi-scaled behavior typical of complex dynamics is provided by a fat-tailed distribution of the waiting time between consecutive jumps. We first argue that CTRW are inadequate to describe macroscopic relaxation processes for three reasons: macroscopic variables are not self-averaging, memory effects require an all-knowing observer, and different mechanisms whereby the jumps affect macroscopic variables all produce identical long time relaxation behaviors. Hence, CTRW shed no light on the link between microscopic and macroscopic dynamics. We then highlight how a more recent approach, Record Dynamics (RD) provides a viable alternative, based on a very different set of physical ideas: while CTRW make use of a renewal process involving identical traps of infinite size, RD embodies a dynamical entrenchment into a hierarchy of traps which are finite in size and possess different degrees of meta-stability. We show in particular how RD produces the stretched exponential, power-law and logarithmic relaxation behaviors ubiquitous in complex dynamics, together with the sub-diffusive time dependence of the Mean Square Displacement characteristic of single particles moving in a complex environment.

Introduction. – Statistical physics is largely about coarse-graining microscopic descriptions into macroscopic ones more closely related to experiments. Thermal relaxation of 'glassy' systems is a case in point: Due to their large number of microscopic configurations from which a deterministic (zero temperature) trajectory never escapes, configuration space can be partitioned into catchments basins which, at finite temperature, retain trajectories for a lapse of time of finite and random duration. We refer to these basins as *traps*, to the time spent in them as *waiting time* and to the transitions between traps as *jumps*. Describing relaxation in terms of traps and jumps greatly reduces the number of variables and constitutes the first step of coarse-graining. Based on *Continuous Time Random Walks* (CTRW) [1–3], a well established approach further assumes that each jump brings the system back to the same situation, i.e. that the sequence of jumps constitutes a *renewal* process. Using a fat-tailed distribution for the waiting time, the multi-scaled relaxation behavior characteristic of complex systems can in many cases be accounted for. Nevertheless, a stationary renewal process is not a natural choice to describe the macroscopic changes occurring in e.g. non-stationary relaxation processes.

As emphasized in the much touted *weak ergodicity breaking* scenario [4, 5] time and ensemble averages differ for renewal processes involving fat-tailed waiting time distributions. This property is closely related to a well-known mathematical result of Sparre-Andersen [6, 7] by the fact that the number of jumps in the interval $[0, t)$ remains a distributed quantity in the limit $t \to \infty$. Hence, in a CTRW description macroscopic quantities have broad distributions even in the thermodynamic limit. A second, related, issue is related to the system size dependence of the average and variance of macroscopic observables. As we argue, both quantities must scale linearly with system size, but fail to do so in CTRW. Thirdly, the memory mechanism implied by CTRW requires an all-knowing observer and, lastly, the long-time tail of the waiting time distribution can hardly be justified in many applications. In summary, even though CTRW appear flexible and emi-

nently applicable, their use to model complex dynamics is a dubious endeavor. We argue below that *Record Dynamics*(RD) is a viable alternative which relies on a completely different physical picture and which avoids the problems affecting CTRW, technically because the jumps are there a Poisson process.

A record in a time series is an entry larger (or smaller) than all the entries that precede it. Records have always been a popular topic, but a recent surge of interest in their statistical properties [8] seems motivated by the ongoing debate on climate change, which is accompanied by a number of record breaking events. That thermal noise records have an impact in complex dynamics was proposed [9] in a model study of Charge Density Waves. Over the years the same idea, which we now refer to as Record Dynamics, has found applications in condensed matter physics [10–13], evolutionary biology [14] and the dynamics of ant societies [15]. The term 'record' in RD signals that overcoming a record-sized dynamical barrier elicits a jump –henceforth in this connection termed *quake*— which brings the system from one trap to a new and previously unexplored trap. RD hence describes a process of *entrenchment* into a hierarchy of traps indexed by dynamical barriers of increasing size [16, 17]. Focusing on the temporal statistics and the macroscopic effects of the quakes, RD provides a coarse-grained description of glassy dynamics.

Critique of CTRW. – The probability $P_{\mathrm{j}}(n,t)$ of n jumps in the time interval $[0,t)$ and its first two moments are discussed below, using the letter s and a superimposed tilde to denote the Laplace variable and the Laplace transform of a function, respectively. Central to the description is the waiting time probability density (PDF) $W(t)$. Whenever its average is finite, the exponential form $W(t)=\exp(-t/t_0)/t_0$ is a natural choice and, we stress, a choice to which our critique does not apply. To model complex relaxation a 'fat-tailed' PDF lacking a finite average

$$W(t)=\frac{\alpha}{t_0}\left(\frac{t}{t_0}\right)^{-\alpha-1}, \quad 0<\alpha<1, \quad t\geq t_0, \qquad (1)$$

is utilized. Through mathematical steps detailed further below, the average and variance of the number of jumps occurring in $(0,t)$ are shown, asymptotically for large t, to be connected by the equation

$$\sigma_{\mathrm{j}}^2(t)\approx\mu_{\mathrm{j}}(t)+\left(\frac{t}{t_0}\right)^{2\alpha}\left(\frac{1}{\alpha\Gamma(2\alpha)}-\frac{1}{\Gamma^2(\alpha+1)}\right), \qquad (2)$$

where Γ is the gamma function. For $\alpha=1$, $\sigma_{\mathrm{j}}^2(t)=\mu_{\mathrm{j}}(t)$. Otherwise, in the large t limit, $\sigma_{\mathrm{j}}^2(t)\propto\mu_{\mathrm{j}}^2(t)$ and since $\sigma_j(t)/\mu_j(t)$ then approaches a constant, the number of jumps retains a broad distribution in the same limit. As the same is true for time averages of quantities subordinated to the jumps but not for the corresponding ensemble averages, ergodicity is 'weakly' broken. In contrast, textbook statistical mechanics teaches us that macroscopic variables are invariably delta-distributed, including cases where broken ergodicity stems from a broken symmetry. To the best of the author's knowledge, no experimental evidence has ever contradicted this result.

Since a single CTRW process cannot consistently describe macroscopic relaxation, let us instead try N independent and simultaneous jumping processes, each supported in one of N domains, a situation typical of spatially extended systems with short-ranged interactions. The average and the variance of the number of jumps throughout the system are in this case both proportional to N, and $\sigma_j(t)/\mu_j(t)\propto N^{-1/2}$ hence vanishes for large N, taking weakly broken ergodicity along. This sounds reassuring, but, as we shall see, the memory behavior implied by the description requires an all-knowing observer.

For any choice of $W(t)$, renewal equations for the jump probability $P_{\mathrm{j}}(n,t)$,

$$P_{\mathrm{j}}(n,t) = \int_0^t P_{\mathrm{j}}(n-1,t')W(t-t')dt' \ ; \quad n=1,2\,. \qquad (3)$$

$$P_{\mathrm{j}}(0,t) = 1-\int_0^t W(t')dt', \qquad (4)$$

are solved in the s domain by

$$\tilde{P}_{\mathrm{j}}(n,s)=\left(\tilde{W}(s)\right)^n\frac{1-\tilde{W}(s)}{s}. \qquad (5)$$

The average number of jumps, $\mu_{\mathrm{j}}(t)=\sum_{k=0}^{\infty}kP_{\mathrm{j}}(k,t)$ and the auxiliary quantity $\mu_{\mathrm{j}^2-\mathrm{j}}(t)=\sum_{k=0}^{\infty}(k^2-k)P_{\mathrm{j}}(k,t)$ have then transforms

$$\tilde{\mu}_{\mathrm{j}}(s)=\frac{\tilde{W}(s)}{s(1-\tilde{W}(s))} \quad \text{and} \quad \tilde{\mu}_{.\mathrm{j}^2-\mathrm{j}}(s)=\frac{2}{s}\left(\frac{\tilde{W}(s)}{1-\tilde{W}(s)}\right)^2, \qquad (6)$$

respectively.

To derive Eq. (2), insert the small s expansion

$$\tilde{W}(s)=1-(t_0 s)^{\alpha}+\mathcal{O}(s^{\alpha+1}) \qquad (7)$$

of the Laplace transform of Eq. (1) into Eq. (5). Inverting the outcome yields

$$\mu_{\mathrm{j}}(t)\approx\frac{1}{\alpha\Gamma(\alpha)}\left(\frac{t}{t_0}\right)^{\alpha} \qquad (8)$$

and

$$\mu_{\mathrm{j}^2-\mathrm{j}}(t)\approx\frac{1}{\alpha\Gamma(2\alpha)}\left(\frac{t}{t_0}\right)^{2\alpha}. \qquad (9)$$

The result follows from $\sigma_{\mathrm{j}}^2(t)=\mu_{\mathrm{j}}(t)+\mu_{\mathrm{j}^2-\mathrm{j}}(t)-\mu_{\mathrm{j}}^2(t)$.

If the jumps constitute the true clock of the dynamics, it is natural to describe their effect on relaxation as a Markov chain. The question is then how the properties of the latter affect the relaxation in the time domain. In general, the propagator of a Markov chain is a linear superposition of exponentially decaying modes, each of the form b^n, where $b<1$. The same is true for averages calculated using the

propagator. Without loss of generality, we now consider the time dependence $g(b,t)$ corresponding to a single mode b^n, which is obtained by averaging n over the probability $P_{\rm j}(n,t)$ that n jumps occur. In the Laplace domain this amounts to

$$\tilde{g}(b,s) = \sum_{n=0}^{\infty} \tilde{P}_{\rm j}(n,s) b^n = \frac{1-\tilde{W}(s)}{s(1-b\tilde{W}(s))}. \quad (10)$$

If $W(t)$ has a finite average t_0, expanding Eq. (10) to lowest order, we find that the mode decays exponentially in time, with a time scale $t_0/(1-b)$ diverging as expected for $b \to 1$. We also note that since $g(b,t)$ actually depends on b, the eigenvalue spectrum of the Markov chain matters in the time domain. This hinges on the s term in the denominator and the $t_0 s$ term in the nominator canceling out. The situation radically differs if $\tilde{W}(s) = 1-(t_0 s)^\alpha$ with $0<\alpha<1$. To leading order, Eq. (10) gives a term proportional to $s^{\alpha-1}$, which in the time domain translates into a power-law decay whose exponent, $-\alpha$, is independent of b. In other words, the value of the exponent α is unrelated to the dynamical effects of the jumps.

Consider now the simple scaling description known as *pure* or *full aging* behavior, which approximately captures some aspects of memory behavior in glassy dynamics. According to pure aging, certain macroscopic variables depend on the ratio $\frac{t}{t_{\rm w}}$, e.g. in the thermoremnant magnetization of spin-glasses [12] , $t > t_{\rm w}$ is the time counted from the initial thermal quench and $t_{\rm w}$ is the time at which the external magnetic field is switched off.

Knowing that the system has remained in the same trap up to time $t_{\rm w}$ at which observations commence, the probability density for exiting the trap at time $t > t_{\rm w}$ is

$$W_R(t_{\rm w},t) = \frac{W(t)}{\int_{t_{\rm w}}^{\infty} W(t')dt'} = \frac{\alpha}{t_{\rm w}}\left(\frac{t}{t_{\rm w}}\right)^{-\alpha-1}, \quad (11)$$

which is identical to the RD expression (17) obtained below by a different route. Since all traps are equivalent in CTRW, the memory behavior implied by Eq.(11) rests on the observer knowing when a trap is entered. This might be experimentally achievable if a single trap describes the whole system, a possibility however already discarded as unphysical. If, however, N independent processes unfold at the same time, the observer must track when every trap is accessed, a task hardly feasible in experiments.

Dynamical hierarchies, records and marginal stability. – Upward rooted binary trees [16, 19–21] whose nodes and height respectively represent traps and their energy provide a convenient coarse-grained representation of energy landscapes with multiple minima. In low temperature thermalization, the 'bottom' states of lowest energy are those mainly occupied, and gaining access to nodes not previously visited entails crossing an energy barrier of record magnitude. Hence, diffusion on a hierarchical structure can be described in terms of RD. In the general case, a record-sized energy fluctuation does not suffice to elicit a quake, simply because there might be no barrier to cross. *Marginal stability* [18] further posits that the barriers successively crossed differ by a minuscule amount. In this limit every record-sized energy fluctuation leads to the crossing of a barrier and record-sized thermal fluctuations trigger quakes. The temporal statistics of the quakes occurring between $t_{\rm w}$ and $t > t_{\rm w}$ is in this limit a Poisson process, whose average $\mu_{\rm q}(t_{\rm w},t) \propto (\ln(t) - \ln(t_{\rm w}))$ is independent of the temperature [9, 14].

To generalize the above results to cases where energy barriers differ by a finite amount, consider that, in general,

$$\mu_{\rm q}(t_{\rm w},t) = \int_{t_{\rm w}}^{t} r(t')dt', \quad (12)$$

where $r(t)$ is the quaking rate and where the form $r(t) = a/t$ corresponds to the logarithmic behavior associated to marginal stability. The generalized form

$$r(t) = at^{x-1}, \quad 0 \le x \le 1, \quad (13)$$

i) reduces to a/t for $x=0$, *ii)* produces time-homogeneous behavior for $x=1$ and, *iii)* integrated with respect to time, yields

$$\mu_{\rm q}(t_{\rm w},t) = a\frac{t^x - t_{\rm w}^x}{x} \stackrel{\rm def}{=} f(t) - f(t_{\rm w}). \quad (14)$$

Since, as later argued, a Poisson distribution still applies, a particle happening to reside in a trap at time $t_{\rm w}$ leaves it at time $t > t_{\rm w}$ with probability

$$P_0(t_{\rm w},t) = \exp(-f(t) + f(t_{\rm w})). \quad (15)$$

In terms of the *lag time* $\Delta = t - t_{\rm w}$ $(0 \le \Delta < \infty)$ commonly used in lieu of t, the residence time R spent in a trap has PDF

$$\begin{aligned} W_R(t_{\rm w},\Delta) &= -\frac{dP_0(t_{\rm w},t_{\rm w}+\Delta)}{d\Delta} \quad (16) \\ &= a\exp[-\frac{a}{x}\left(t_{\rm w}+\Delta\right)^x - t_{\rm w}^x)]\left(t_{\rm w}+\Delta\right)^{-1+x}. \end{aligned}$$

We note in passing that Δ is often denoted by τ or by t in the literature, both usages unfortunately clashing with our present notation. For $x \ll 1$, one obtains

$$W_R(t_{\rm w},\Delta) \approx \frac{a}{t_{\rm w}}\left(1+\frac{\Delta}{t_{\rm w}}\right)^{-a-1} \quad (17)$$

which is equivalent to the CTRW expression given by Eq. (1). Importantly, the time scale parameter which is fixed in CTRW is simply the system age in RD. Secondly, Eq. (17) contains a stretched exponential, and its similarity to $W(t)$ is restricted to the limit $x \to 0$. Thirdly, the RD parameter a is positive but not *a priori* limited to the unit interval. For a single hopping process and in the limit $x \to 0$, if each barrier record triggers a quake, $a = 1$, otherwise $0 < a < 1$. In extended systems, where several independent hopping processes occur simultaneously, a is proportional to the size of the system, as we later argue.

According to Eq. (17), the average or characteristic time spent in a trap occupied (but *not* necessarily entered) at time t_{w} is

$$t_0(t_{\mathrm{w}}) = a^{-1}\left(\frac{x}{a}\right)^{\frac{1}{x}-1} \exp(\frac{a}{x}t_{\mathrm{w}}^x)\Gamma(\frac{1}{x}, \frac{a}{x}t_{\mathrm{w}}^x), \qquad (18)$$

where $\Gamma(s,z) = \int_z^\infty \exp(-y)y^{s-1}dy$ is the incomplete upper gamma function. As a check we note that $t_0(t_{\mathrm{w}}) = a^{-1}$ for $x = 1$ and that in the limit $x \to 0$ $t_0(t_{\mathrm{w}}) \to t_{\mathrm{w}}$, for $a > 1$. In the same limit and for $a \leq 1$, the average is infinite, but t_{w} still provides the characteristic time scale for the power-law decay implied by Eq. (17).

Assume now that an application specific function f has been found such that the probability density for the occurrence of a quake is uniform in the stretched observation interval $f(t) - f(t_{\mathrm{w}})$. Partitioning the interval into M subintervals, let p be the probability that a quake falls into any of these and note that the probability for n quakes occurring is the binomial $B(p, n, M)$. In the relevant limit $p \to 0$, $M \to \infty$ and $pM \to \mu_{\mathrm{q}}$, the binomial tends to a Poisson distribution, as claimed.

Using a binary tree to coarse-grain an energy landscape [16], we just argued that RD dynamics arises in two ways: in the limit $x \to 0$, successive barriers increase marginally, records in the impinging noise induce barrier crossings and, on average, the typical number $n \approx \mu_{\mathrm{q}}$ of barriers crossed at time t is proportional to $\ln(t)$. The Arrhenius relation $\ln(t) \propto b(n)/T$ where T is the temperature and $b(n)$ is the height of the $n'th$ barrier then implies $b(n) \propto Tn$. If marginal stability is relinquished, i.e for $x > 0$, we find $\ln(\mu_{\mathrm{q}}) \approx \ln(n(t)) \propto x\ln(t)$ for $t >> t_{\mathrm{w}}$, from which we infer that the size of the $n'th$ barrier crossed is $b(n) \propto (T/x)\ln(n)$.

The time dependence of a macroscopic quantity, say g, is calculated in RD by averaging its dependence $\tilde{g}(n)$ over the probability of n quakes occurring in (t_{w}, t), i.e.

$$g(t_{\mathrm{w}}, t) = e^{-\mu_{\mathrm{q}}(t_{\mathrm{w}},t)} \sum_{n=0}^{\infty} \tilde{g}(n) \frac{(\mu_{\mathrm{q}}(t_{\mathrm{w}}, t))^n}{n!}, \qquad (19)$$

where $\mu_q(t_{\mathrm{w}}, t)$ is given in Eq. (14). As a first example, assume $\tilde{g}(n) = c(n = 0)b^n$, where c expresses the initial condition and where $b < 1$. The stretched exponential behavior ubiquitous in glassy dynamics [22, 23]

$$g(t_{\mathrm{w}}, t) = c(t_{\mathrm{w}})e^{-\mu_q(t_{\mathrm{w}},t)(1-b)} = c(t_{\mathrm{w}})e^{-(t^x - t_{\mathrm{w}}^x)\frac{a(1-b)}{x}}, \qquad (20)$$

immediately follows. If g is a one-point average, $c(t_{\mathrm{w}}) = c(t_0)e^{-t_{\mathrm{w}}^x \frac{a(1-b)}{x}}$ and there is in reality only one time argument t. In contrast, a two-point correlation function with $c(t_{\mathrm{w}} = 1)$ truly depends on two arguments, as well known.

Again using the lag time $\Delta = t - t_{\mathrm{w}}$ Eq. (20) is recast, for $\Delta/t_{\mathrm{w}} \ll 1$ into

$$g(t_{\mathrm{w}}, \Delta) = c(t_{\mathrm{w}})e^{-\frac{\Delta}{\tau(t_{\mathrm{w}})}}, \qquad (21)$$

an exponential decay with a characteristic time constant $\tau(t_{\mathrm{w}}) = \frac{xt_{\mathrm{w}}^{1-x}}{a(1-b)}$. A relaxation time increasing with system age is experimentally observed in colloidal systems [22,24]. The age dependence of the life-time of the exponential approximation given in Eq. (21) is not usually discussed, but follows nevertheless by the simple Taylor expansion given above. In the limit $x \to 0$, Eq. (20) reduces to the power-law

$$g(t_{\mathrm{w}}, t) = c(t_{\mathrm{w}})\left(\frac{t}{t_{\mathrm{w}}}\right)^{-a(1-b)} \approx c(t_{\mathrm{w}})\exp(-\frac{a(1-b)}{t_{\mathrm{w}}}\Delta), \qquad (22)$$

where the exponential approximation holds for $\Delta \ll t_{\mathrm{w}}$. Anticipating a later observation, we now let μ_{q} be proportional to the system size N of a macroscopic system via $a = N\tilde{a}$, where $\tilde{a}$ is a new constant. Secondly, we treat $b^n = \exp(\tilde{b}n)$ as one mode of a relaxation process parameterized by n in lieu of time. Of the N eigenvalues in the spectrum most will approach zero as $N \to \infty$. A glance at Eq. (20) shows that only those for which $\tilde{b} = \mathcal{O}(1/N)$ produce a macroscopic decay independent of N. If the decay of $\tilde{b}$ with N is faster respectively slower than $1/N$, the corresponding mode in the time domain either has a 'frozen' constant value or immediately decays to zero in the large N limit. Note that the stretching exponent x is independent of system size, while the exponent $-a(1-b)$ in Eq. (22) is only N independent if, as just discussed, $\tilde{b} = \mathcal{O}(1/N)$.

In summary, simple and general RD arguments lead to dynamical behaviors common in complex systems: stretched exponential relaxation and power laws with pure aging scaling. The sub-diffusive behavior of a single particle moving in a complex environment is discussed next.

Subdiffusion. – Irreversible single particle jumps in complex environments, e.g. binary Lennard-Jones mixtures in their glassy phase [25] are indicative of collective configurational re-arrangements. The same is, we believe, true for single particle diffusion in a living cell, a problem which has recently been modeled using CTRW [26]. It is difficult to imagine how a living cell can contain the traps of infinite, or at least very large, spatial extension needed to produce a waiting time distribution with a long-time tail, especially considering that the diffusing particle and its enclosure have similar length scales.

Experimental data for dense colloidal system [27] reanalyzed in [13] show that single particle Mean Square Displacement (MSD) grow logarithmically, a property explained in Ref. [13] using RD. This result, which corresponds to the limit $x \to 0$ in Eq. (24), suggests that single particles in general probe the local re-arrangements of their aging environment. This leads to sub-diffusion formulas rather similar to their CTRW counterparts. Distinguishing between the two approaches can therefore be experimentally challenging, as it e.g. requires an analysis of higher moments and/or an explicit investigation of age dependencies via ensemble averages. To avoid convoluted typography the same symbol is used for the moments of the particle position, irrespective of the method used.

Note however that CTRW formulas have one time argument, while RD formulas mostly have two.

After performing n independent jumps, each associated to a random additive position change Δx_i, a point particle is located at

$$X(n) = \sum_{i=1}^{n} \Delta x_i. \quad (23)$$

Assume for simplicity that the identically distributed Δx_i have vanishing odd moments and denote their second and fourth central moments by e_2 and e_4, respectively. The form of these moments will depend on e.g. whether the particles move in a potential well, but the arguments below do not.

After n jumps, the variance of the particle position or, equivalently its MSD, is $\sigma_X^2(n) = ne_2$. Hence,

$$\sigma_X^2(t) = \mu_{\rm j}(t)e_2 \quad \text{and} \quad \sigma_X^2(t_{\rm w}, t) = \mu_{\rm q}(t_{\rm w}, t)e_2 \quad (24)$$

for CTRW and for RD, respectively. Explicitly, using Eq. (14), we find the sub-diffusive behavior

$$\sigma_X^2(t_{\rm w}, t) = a\frac{t^x - t_{\rm w}^x}{x}e_2. \quad (25)$$

Note that if the first jump moment e_1 differs from zero a formula of the same type holds for the average position. Writing for convenience $t_{\rm w} = y$ and $t = y + \Delta$, where Δ is the lag time, and expanding Eq. (14) to first order in Δ, we find

$$\mu_{\rm q}(y, y+\Delta) \approx \frac{a}{x^2}\frac{d(y^x)}{dy}\Delta. \quad (26)$$

Experimentally, the variance is estimated using the time integral

$$\overline{\sigma_X^2}(t) = \frac{1}{t_{\max} - \Delta}\int_0^{t_{\max}-\Delta} \left[X(y+\Delta) - X(y)\right]^2 dy, \quad (27)$$

where $t_{\max}$ is the largest observation time. This corresponds to averaging $\mu_{\rm q}(y, y+\Delta)$ with respect to y over the same time span. To first order in Δ, the time averaged particle MSD is then

$$\overline{\sigma_X^2}(\Delta, t_{\max}) \approx \frac{a}{x^2}t_{\max}^{x-1}\Delta \quad \text{for} \quad \Delta < t_{\max} \quad (28)$$

If, on the other hand, $\Delta \approx t_{\max}$, time averaging is of dubious value, and Eq. (24) directly implies

$$\sigma_X^2(\Delta) \approx \frac{a}{x}\Delta^x. \quad (29)$$

Taken together, Eqs. (28) and (29) describe a cross-over of the MSD from a linear to a sub-linear time dependence, a behavior observed by Jeon et al. [26] in their experiments on lipid granules in an intracellular environment. These authors claim that their findings 'unanimously' point to CTRW as the mechanism behind sub-diffusive behavior, but as we just argued RD offers an alternative explanation.

To better discriminate between CTRW and RD, consider the ratio B between the fourth and the squared second moment of X. Given n jumps, the fourth moment is

$$\mu_{X^4}(n) = ne_4 + (n^2 - n)e_2^2. \quad (30)$$

Correspondingly in the time domain

$$\mu_{X^4}(t) \approx \mu_{\rm j}(t)e_4 + \left(\frac{t}{t_0}\right)^{2\alpha}\frac{e_2^2}{\alpha\Gamma(2\alpha)} \quad (31)$$

for CTRW and

$$\mu_{X^4}(t_{\rm w}, t) = \mu_{\rm q}(t_{\rm w}, t)e_4 + (\mu_{\rm q}(t_{\rm w}, t))^2 e_2^2 \quad (32)$$

for RD. For CTRW, the ratio

$$B(t) = \frac{\mu_{X^4}(t)}{(\sigma_X^2(t))^2} \approx \frac{\alpha\Gamma^2(\alpha)}{\Gamma(2\alpha)} + \frac{e_4}{e_2^2}\frac{1}{\mu_n(t)} \quad (33)$$

approaches $\frac{\alpha\Gamma^2(\alpha)}{\Gamma(2\alpha)}$ as $t \to \infty$. In the same limit, the RD expression

$$B(t_{\rm w}, t) = 1 + \frac{e_4}{e_2^2}\frac{1}{\mu_{\rm q}(t_{\rm w}, t)} \quad (34)$$

approaches unity, independently of the exponent x. This difference offers an opportunity to discriminate between the two descriptions. Assuming that a salient event defining the age of the system can be identified, a second possibility is to investigate whether the particle MSD has an aging dependence by performing ensemble averages. This dependence is present in RD but not in CTRW.

Discussion. – The eminent applicability of CTRW conceals a number of theoretical issues. Firstly, fat-tailed waiting time PDFs for spatially confined processes, such as diffusion in cellular environments are in general curtailed by finite size effects. Secondly, macroscopic variables modeled with a single CTRW feature an unphysical lack of self-averaging. Finally, since memory in CTRW cannot be rooted in the unchanging physical properties of the traps visited, it must be rooted in the observer's awareness of the time at which a trap is entered. This knowledge is only available (in principle) if traps pertain to the entire system, the possibility already invalidated by the lack of self-averaging.

Broadly speaking, RD has the same range of applications as CTRW, but shares none of their problems: Residence times have, with a single exception, a finite average which increases systematically with system age. This means that, in contrast to CTRW, macroscopic configurations contain information on the system' s age, a fact which naturally explains memory behavior in RD. Since quakes are a Poisson process, albeit of an unusual kind, subordinated physical quantities are always self-averaging. Using averages over the number of quakes, RD produces a wide ranging analytical description of glassy relaxation and of single particle diffusion in complex environments.

A hierarchical configuration space structure which now seems to find its way into macroeconomics [29], was advocated long ago by H. Simon [28] as a defining property of complexity. Whenever such description applies, crossing record sized barriers triggers quakes. Conversely, analyzing the dynamical effects of record sized perturbations on the stability of a system, a procedure which can in principle be purely observational, provides important clues on the configuration space structure. This line of investigation has great potential interest in complex dynamics and can benefit from a recent considerable interest in record statistics [8].

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